

# GENERALISED BIALGEBRAS AND ENTWINED MONADS AND COMONADS

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**ABSTRACT.** Jean-Louis Loday has defined generalised bialgebras and proved structure theorems in this setting which can be seen as general forms of the Poincaré-Birkhoff-Witt and the Cartier-Milnor-Moore theorems. It was observed by the present authors that parts of the theory of generalised bialgebras are special cases of results on entwined monads and comonads and the corresponding mixed bimodules. In this article the Rigidity Theorem of Loday is extended to this more general categorical framework.

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## 1. INTRODUCTION

The introduction of *entwining structures* between an algebra and a coalgebra by T. Brzeziński and S. Majid in [2] opened new perspectives in the mathematical treatment of quantum principal bundles. It turned out that these structures are special cases of distributive laws treated in Beck's paper [1]. The latter were also used by Turi and Plotkin [15] in the context of operational semantics.

These observations led to a revival of the investigation of various forms of distributive laws. In a series of papers [12, 13, 14] it was shown how they allow for formulating the theory of Hopf algebras and Galois extensions in a general categorical setting.

On the other hand, *generalised bialgebras* as defined in Loday [7, Section 2.1], are vector spaces which are algebras over an *operad*  $\mathcal{A}$  and coalgebras

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over a cooperad  $\mathcal{C}$ . Moreover, the operad  $\mathcal{A}$  and the cooperad  $\mathcal{C}$  are required to be related by a distributive law. Since any operad  $\mathcal{A}$  yields a monad  $\mathcal{T}_{\mathcal{A}}$  and  $\mathcal{A}$ -algebras are nothing else than  $\mathcal{T}_{\mathcal{A}}$ -modules, and similarly any cooperad  $\mathcal{C}$  yields a comonad  $\mathcal{G}_{\mathcal{C}}$  and  $\mathcal{C}$ -coalgebras are nothing else than  $\mathcal{G}_{\mathcal{C}}$ -comodules, generalised bialgebras have interpretations in terms of bimodules over a bimonad in the sense of [13].

The purpose of the present paper is to make this relationships more precise (as proposed in [13, 2.3]). We provide a theory for functors on fairly general categories which leads to the Rigidity Theorem [7, 2.5.1] as a special case. The details of this application are described in Section 6.

## 2. COMODULES AND ADJOINT FUNCTORS

In this section we provide basic notions and properties of comodule functors and adjoint pairs of functors. Throughout the paper  $\mathbb{A}$  and  $\mathbb{B}$  will denote any categories.

**2.1. Monads and comonads.** Recall that a *monad*  $\mathcal{T}$  on  $\mathbb{A}$  is a triple  $(T, m, e)$  where  $T : \mathbb{A} \rightarrow \mathbb{A}$  is a functor with natural transformations  $m : TT \rightarrow T$ ,  $e : 1 \rightarrow T$  satisfying associativity and unitality conditions. A  $\mathcal{T}$ -*module* is an object  $a \in \mathbb{A}$  with a morphism  $h : T(a) \rightarrow a$  subject to associativity and unitality conditions. The (Eilenberg-Moore) category of  $\mathcal{T}$ -modules is denoted by  $\mathbb{A}_{\mathcal{T}}$  and there is a free functor

$$\phi_{\mathcal{T}} : \mathbb{A} \rightarrow \mathbb{A}_{\mathcal{T}}, a \mapsto (T(a), m_a),$$

which is left adjoint to the forgetful functor

$$U_{\mathcal{T}} : \mathbb{A}_{\mathcal{T}} \rightarrow \mathbb{A}, (a, h) \mapsto a.$$

Dually, a *comonad*  $\mathcal{G}$  on  $\mathbb{A}$  is a triple  $(G, \delta, \varepsilon)$  where  $G : \mathbb{A} \rightarrow \mathbb{A}$  is a functor with natural transformations  $\delta : G \rightarrow GG$ ,  $\varepsilon : G \rightarrow 1$ , and  $\mathcal{G}$ -*comodules* are objects  $a \in \mathbb{A}$  with morphisms  $\theta : a \rightarrow G(a)$ . Both notions are subject to coassociativity and counitality conditions. The (Eilenberg-Moore) category of  $\mathcal{G}$ -comodules is denoted by  $\mathbb{A}^{\mathcal{G}}$  and there is a cofree functor

$$\phi^{\mathcal{G}} : \mathbb{A} \rightarrow \mathbb{A}^{\mathcal{G}}, a \mapsto (G(a), \delta_a),$$

which is right adjoint to the forgetful functor

$$U^{\mathcal{G}} : \mathbb{A}^{\mathcal{G}} \rightarrow \mathbb{A}, (a, \theta) \mapsto a.$$

**2.2.  $\mathcal{G}$ -comodule functors.** For a comonad  $\mathcal{G} = (G, \delta, \varepsilon)$  on  $\mathbb{A}$ , a functor  $F : \mathbb{B} \rightarrow \mathbb{A}$  is a *left  $\mathcal{G}$ -comodule* if there exists a natural transformation  $\alpha_F : F \rightarrow GF$  inducing commutativity of the diagrams

$$\begin{array}{ccc} F & \xrightarrow{\alpha_F} & GF \\ \searrow & & \downarrow \varepsilon F \\ & & F, \end{array} \quad \begin{array}{ccc} F & \xrightarrow{\alpha_F} & GF \\ \alpha_F \downarrow & & \downarrow \delta F \\ GF & \xrightarrow{G\alpha_F} & GGF. \end{array}$$

Symmetrically, one defines right  $\mathcal{G}$ -comodules.

**2.3.  $\mathcal{G}$ -comodules and adjoint functors.** Consider a comonad  $\mathcal{G} = (G, \delta, \varepsilon)$  on  $\mathbb{A}$  and an adjunction  $F \dashv R : \mathbb{A} \rightarrow \mathbb{B}$  with counit  $\sigma : FR \rightarrow 1$ .

There exist bijective correspondences (see [3]) between:

- functors  $K : \mathbb{B} \rightarrow \mathbb{A}^{\mathcal{G}}$  with commutative diagrams

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{K} & \mathbb{A}^{\mathcal{G}} \\ & \searrow F & \downarrow U^{\mathcal{G}} \\ & & \mathbb{A}; \end{array}$$

- left  $\mathcal{G}$ -comodule structures  $\alpha_F : F \rightarrow GF$  on  $F$ ;
- comonad morphisms from the comonad generated by the adjunction  $F \dashv R$  to the comonad  $\mathcal{G}$ ;
- right  $\mathcal{G}$ -comodule structures  $\beta_R : R \rightarrow RG$  on  $R$ .

In this case,  $K(b) = (F(b), \alpha_b)$  for some morphism  $\alpha_b : F(b) \rightarrow GF(b)$ , and the collection  $\{\alpha_b, b \in \mathbb{B}\}$  constitutes a natural transformation  $\alpha_F : F \rightarrow GF$  making  $F$  a  $\mathcal{G}$ -comodule. Conversely, if  $(F, \alpha_F : F \rightarrow GF)$  is a  $\mathcal{G}$ -module, then  $K : \mathbb{B} \rightarrow \mathbb{A}^{\mathcal{G}}$  is defined by  $K(b) = (F(b), (\alpha_F)_b)$ .

For any left  $\mathcal{G}$ -comodule structure  $\alpha_F : F \rightarrow GF$ , the composite

$$t : FR \xrightarrow{\alpha_F R} GFR \xrightarrow{G\sigma} G$$

is a comonad morphism from the comonad generated by the adjunction  $F \dashv R$  to the comonad  $\mathcal{G}$ . Then the corresponding right  $\mathcal{G}$ -comodule structure  $\beta_R : R \rightarrow RG$  on  $R$  is the composite  $R \xrightarrow{\eta^R} RFR \xrightarrow{Rt} RG$ .

Conversely, given a right  $\mathcal{G}$ -comodule structure  $\beta_R : R \rightarrow RG$  on  $R$ , then the comonad morphism  $t : FR \rightarrow \mathcal{G}$  is the composite

$$FR \xrightarrow{F\beta_R} FRG \xrightarrow{\sigma G} G,$$

while the corresponding left  $\mathcal{G}$ -comodule structure  $\alpha_F : F \rightarrow GF$  on  $F$  is the composite  $F \xrightarrow{F\eta} FRF \xrightarrow{tF} GF$ .

We need the following result, the dual version of Dubuc's theorem [4].

**2.4. Dubuc's Adjoint Triangle Theorem.** For categories  $\mathbb{A}$ ,  $\mathbb{B}$  and  $\mathbb{C}$ , let  $F : \mathbb{A} \rightarrow \mathbb{B}$  be a functor with right adjoint  $U$  with unit  $\eta : 1 \rightarrow UF$ , and let  $K : \mathbb{C} \rightarrow \mathbb{A}$  be such that  $F' = FK : \mathbb{C} \rightarrow \mathbb{B}$  has a right adjoint with counit  $\varepsilon' : F'U' \rightarrow 1$ . Define

$$\alpha : KU' \xrightarrow{\eta KU'} UFKU' = UF'U' \xrightarrow{U\varepsilon'} U.$$

If  $\mathbb{C}$  has equalisers of coreflexive pairs and the functor  $F$  is of descent type, then  $K$  has a right adjoint  $R$  which can be calculated as the equaliser

$$\begin{array}{ccc} R & \longrightarrow & U'F \\ & \searrow \eta' U'F & \xrightarrow{U'F\eta} U'FUF \\ & & \nearrow U'F\alpha F \\ & & U'F'U'F = U'FKU'F. \end{array}$$

**2.5. Right adjoint of  $K$ .** Now fix a functor  $K : \mathbb{B} \rightarrow \mathbb{A}^{\mathcal{G}}$  with  $U^{\mathcal{G}}K = F$  and suppose that the category  $\mathbb{B}$  has equalisers of coreflexive pairs. It then follows from Theorem 2.4 that the functor  $K$  has a right adjoint  $\overline{R}$  which is determined by the equaliser diagram

$$(2.1) \quad \overline{R} \xrightarrow{i} RU^{\mathcal{G}} \xrightleftharpoons[\beta_R U^{\mathcal{G}}]{RU^{\mathcal{G}}\eta^{\mathcal{G}}} RGU^{\mathcal{G}} = RU^{\mathcal{G}}\phi^{\mathcal{G}}U^{\mathcal{G}},$$

where  $\eta^{\mathcal{G}} : 1 \rightarrow \phi^{\mathcal{G}}U^{\mathcal{G}}$  is the unit of the adjunction  $U^{\mathcal{G}} \dashv \phi^{\mathcal{G}}$ .

An easy inspection shows that the value of  $\overline{R}$  at  $(a, \theta) \in \mathbb{A}^{\mathcal{G}}$  is given by the equaliser diagram

$$(2.2) \quad \overline{R}(a, \theta) \xrightarrow{i_{(a, \theta)}} R(a) \xrightleftharpoons[(\beta_R)_a]{R(\theta)} RG(a).$$

**2.6. Theorem.** (see [10, Theorem 4.4]) *A functor  $K : \mathbb{B} \rightarrow \mathbb{A}^{\mathcal{G}}$  with  $U^{\mathcal{G}}K = F$  is an equivalence of categories if and only if*

- (i) *the functor  $F$  is comonadic, and*
- (ii)  *$t_K$  is an isomorphism of comonads.*

### 3. DISTRIBUTIVE LAWS

Distributive laws were introduced by Beck in [1]. Here we are mainly interested in the following case (e.g. [5] or [16, 5.3]).

**3.1. Mixed distributive laws.** Let  $\mathcal{T} = (T, m, e)$  be a monad and  $\mathcal{G} = (G, \delta, \varepsilon)$  a comonad on the category  $\mathbb{A}$ . A natural transformation

$$\lambda : TG \rightarrow GT$$

is said to be a *mixed distributive law* or a *(mixed) entwining* provided it induces commutative diagrams

$$\begin{array}{ccc} TTG & \xrightarrow{mG} & TG \\ T\lambda \downarrow & & \downarrow \lambda \\ TGT & \xrightarrow{\lambda T} GTT & \xrightarrow{Gm} GT, \end{array} \quad \begin{array}{ccc} TG & \xrightarrow{T\delta} TGG & \xrightarrow{\lambda G} GTG \\ \lambda \downarrow & & \downarrow G\lambda \\ GT & \xrightarrow{\delta T} & GGT, \end{array}$$

$$\begin{array}{ccc}
 G & \xrightarrow{eG} & TG \\
 & \searrow Ge & \downarrow \lambda \\
 & & GT,
 \end{array}
 \quad
 \begin{array}{ccc}
 TG & \xrightarrow{T\varepsilon} & T \\
 \downarrow \lambda & & \nearrow \varepsilon T \\
 GT. & & 
 \end{array}$$

Recall (for example, from [17]) that if  $\mathcal{T}$  is a monad and  $\mathcal{G}$  is a comonad on a category  $\mathbb{A}$ , then the following structures are in bijective correspondence:

- mixed distributive laws  $\lambda : TG \rightarrow GT$ ;
- comonads  $\hat{\mathcal{G}} = (\hat{G}, \hat{\delta}, \hat{\varepsilon})$  on  $\mathbb{A}_{\mathcal{T}}$  that extend  $\mathcal{G}$  in the sense that  $U_{\mathcal{T}}\hat{G} = GU_{\mathcal{T}}$ ,  $U_{\mathcal{T}}\hat{\varepsilon} = \varepsilon U_{\mathcal{T}}$  and  $U_{\mathcal{T}}\hat{\delta} = \delta U_{\mathcal{T}}$ ;
- monads  $\hat{\mathcal{T}} = (\hat{T}, \hat{m}, \hat{e})$  on  $\mathbb{A}^{\mathcal{G}}$  that extend  $\mathcal{T}$  in the sense that  $U^{\mathcal{G}}\hat{T} = TU^{\mathcal{G}}$ ,  $U^{\mathcal{G}}\hat{e} = eU^{\mathcal{G}}$  and  $U^{\mathcal{G}}\hat{m} = mU^{\mathcal{G}}$ .

Recall also that

$$\begin{aligned}
 \hat{G}(a, h) &= (G(a), G(h) \cdot \lambda_a), \quad \hat{\varepsilon}_{(a, h)} = \varepsilon_a, \quad \hat{\delta}_{(a, h)} = \delta_a, \quad \text{for any } (a, h) \in \mathbb{A}_{\mathcal{T}}; \\
 \hat{T}(a, \theta) &= (T(a), \lambda_a \cdot T(\theta)), \quad \hat{e}_{(a, \theta)} = e_a, \quad \hat{m}_{(a, \theta)} = m_a \quad \text{for any } (a, \theta) \in \mathbb{A}^{\mathcal{G}}.
 \end{aligned}$$

It follows that for a mixed distributive law  $\lambda : TG \rightarrow GT$  one may assume

$$(\mathbb{A}^{\mathcal{G}})_{\hat{\mathcal{T}}} = (\mathbb{A}_{\mathcal{T}})^{\hat{\mathcal{G}}}.$$

We write  $\mathbb{A}_{\mathcal{T}}^{\hat{\mathcal{G}}}(\lambda)$  for this category, whose objects, called *TG-bimodules* in [5], are triples  $(a, h, \theta)$ , where  $(a, h) \in \mathbb{A}_{\mathcal{T}}$ ,  $(a, \theta) \in \mathbb{A}^{\mathcal{G}}$  with commuting diagram

$$\begin{array}{ccccc}
 T(a) & \xrightarrow{h} & a & \xrightarrow{\theta} & G(a) \\
 T(\theta) \downarrow & & & & \uparrow G(h) \\
 TG(a) & \xrightarrow{\lambda_a} & GT(a) & & 
 \end{array}$$

Morphisms in this category are morphisms in  $\mathbb{A}$  which are  $\mathcal{T}$ -module as well as  $\mathcal{G}$ -comodule morphisms.

**3.2. Entwined monads and comonads.** Let  $\mathcal{T} = (T, m, e)$  be a monad,  $\mathcal{G} = (G, \delta, \varepsilon)$  a comonad on  $\mathbb{A}$ , and consider an entwining  $\lambda : TG \rightarrow GT$  from  $\mathcal{T}$  to  $\mathcal{G}$ . Denote by  $\hat{\mathcal{T}} = (\hat{T}, \hat{m}, \hat{e})$  the monad on  $\mathbb{A}^{\mathcal{G}}$  lifting  $\mathcal{T}$  and by  $\hat{\mathcal{G}} = (\hat{G}, \hat{\delta}, \hat{\varepsilon})$  the comonad on  $\mathbb{A}_{\mathcal{T}}$  lifting  $\mathcal{G}$ .

Suppose there exists a functor  $K : \mathbb{A} \rightarrow (\mathbb{A}_{\mathcal{T}})^{\hat{\mathcal{G}}}$  with commutative diagram

$$(3.1) \quad \begin{array}{ccc}
 \mathbb{A} & \xrightarrow{K} & (\mathbb{A}_{\mathcal{T}})^{\hat{\mathcal{G}}} \\
 & \searrow \phi_{\mathcal{T}} & \downarrow U^{\hat{\mathcal{G}}} \\
 & & \mathbb{A}_{\mathcal{T}}
 \end{array}$$

and consider the corresponding right  $\hat{\mathcal{G}}$ -comodule structure on  $U_{\mathcal{T}}$  (see 2.3)

$$\beta = \beta_{U_{\mathcal{T}}} : U_{\mathcal{T}} \rightarrow U_{\mathcal{T}}\hat{G} = GU_{\mathcal{T}}.$$

Then, for any  $(a, h) \in \mathbb{A}_{\mathcal{T}}$ , the  $(a, h)$ -component  $\beta_{(a, h)} = (\beta_{U_{\mathcal{T}}})_{(a, h)}$  of  $\beta$  is a morphism  $a \rightarrow G(a)$  in  $\mathbb{A}$ . Assuming that  $\mathbb{A}$  admits coreflexive equalisers, we obtain by (2.2) that the functor  $K$  admits a right adjoint  $R$  whose value at  $((a, h), \theta) \in (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}$  appears as the equaliser

$$(3.2) \quad R((a, h), \theta) \xrightarrow{i_{((a, h), \theta)}} a \underset{\beta_{(a, h)}}{\overset{\theta}{\rightrightarrows}} G(a).$$

Consider now the left  $\widehat{\mathcal{G}}$ -comodule structure  $\alpha = \alpha_{\phi_{\mathcal{T}}} : \phi_{\mathcal{T}} \rightarrow \widehat{\mathcal{G}}\phi_{\mathcal{T}}$  on  $\phi_{\mathcal{T}}$  induced by the commutative diagram (3.1). As shown in [12, Theorem 2.4], for any  $(a, h) \in \mathbb{A}_{\mathcal{T}}$ , the component  $(t_K)_{(a, h)}$  of the comonad morphism  $t_K : \phi_{\mathcal{T}}U_{\mathcal{T}} \rightarrow \widehat{\mathcal{G}}$ , corresponding to the diagram (3.1), is the composite

$$(3.3) \quad T(a) \xrightarrow{T(\beta_{(a, h)})} TG(a) \xrightarrow{\lambda_a} GT(a) \xrightarrow{G(h)} G(a).$$

#### 4. GROUPLIKE MORPHISMS

Let  $\mathcal{G} = (G, \delta, \varepsilon)$  be a comonad on a category  $\mathbb{A}$ . By [12, Definition 3.1], a natural transformation  $g : 1 \rightarrow G$  is called a *grouplike morphism* provided it is a comonad morphism from the identity comonad to  $\mathcal{G}$ , that is, it induces commutative diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{g} & G \\ & \searrow & \downarrow \varepsilon \\ & = & 1, \end{array} \quad \begin{array}{ccc} 1 & \xrightarrow{g} & G \\ & \searrow & \downarrow \delta \\ & gg & GG. \end{array}$$

The dual notion is that of *augmentation*. A monad  $\mathcal{T}$  on  $\mathbb{A}$  has an augmentation if it is endowed with a monad morphism  $T \rightarrow 1$ .

Let  $\mathcal{T} = (T, m, e)$  be a monad and  $\mathcal{G} = (G, \delta, \varepsilon)$  a comonad on  $\mathbb{A}$  with an entwining  $\lambda : TG \rightarrow GT$ . If  $\mathcal{G}$  has a grouplike morphism  $g : 1 \rightarrow G$ , then the above conditions guarantee that the morphisms  $(g_a : a \rightarrow G(a))_{(a, h) \in \mathbb{A}_{\mathcal{T}}}$  form the components of a right  $\widehat{\mathcal{G}}$ -comodule structure  $\beta = \beta_{U_{\mathcal{T}}} : U_{\mathcal{T}} \rightarrow U_{\mathcal{T}}\widehat{\mathcal{G}}$  on the functor  $U_{\mathcal{T}} : \mathbb{A}_{\mathcal{T}} \rightarrow \mathbb{A}$ .

Observing that in the diagram

$$\begin{array}{ccccccc} T(a) & \xrightarrow{T(g_a)} & TG(a) & \xrightarrow{\lambda_a} & GT(a) & & \\ T(e_a) \downarrow & & TG(e_a) \downarrow & & GT(e_a) \downarrow & \searrow & \\ TT(a) & \xrightarrow{T(g_{T(a)})} & TGT(a) & \xrightarrow{\lambda_{T(a)}} & GTT(a) & \xrightarrow{G(m_a)} & GT(a) \end{array}$$

- the left hand square commutes by naturality of  $g$ ,
- the right hand square commutes by naturality of  $\lambda$ , and
- the triangle commutes since  $e$  is the unit for the monad  $\mathcal{T}$ ,

and recalling that  $\alpha$  is the composite  $\phi_{\mathcal{T}} \xrightarrow{\phi_{\mathcal{T}}\eta_{\mathcal{T}}} \phi_{\mathcal{T}}U_{\mathcal{T}}\phi_{\mathcal{T}} \xrightarrow{t_K\phi_{\mathcal{T}}} \widehat{G}\phi_{\mathcal{T}}$ , one concludes by (3.3) that

$$(4.1) \quad \text{for every } a \in \mathbb{A}, \quad \alpha_a = \lambda_a \cdot T(g_a).$$

This leads to a functor

$$K_g : \mathbb{A} \rightarrow (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}, \quad a \mapsto ((T(a), m_a), \lambda_a \cdot T(g_a)),$$

and the commutative diagram

$$(4.2) \quad \begin{array}{ccc} \mathbb{A} & \xrightarrow{K_g} & (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}} \\ & \searrow \phi_{\mathcal{T}} & \downarrow U^{\widehat{\mathcal{G}}} \\ & & \mathbb{A}_{\mathcal{T}}. \end{array}$$

In this case we say that the comparison functor  $K_g$  is induced by the grouplike morphism  $g : 1 \rightarrow G$ .

Specialising now Theorem 2.6 to the present situation gives

**4.1. Theorem.** *Let  $\mathcal{T} = (T, m, e)$  be a monad and  $\mathcal{G} = (G, \delta, \varepsilon)$  a comonad on  $\mathbb{A}$  with an entwining  $\lambda : TG \rightarrow GT$ . If  $g : 1 \rightarrow G$  is a grouplike morphism of the comonad  $\mathcal{G}$ , then the induced functor  $K_g : \mathbb{A} \rightarrow (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}$  is an equivalence of categories if and only if*

- (i) *the functor  $\phi_{\mathcal{T}}$  is comonadic, and*
- (ii) *the composite*

$$(4.3) \quad T(a) \xrightarrow{T(g_a)} TG(a) \xrightarrow{\lambda_a} GT(a) \xrightarrow{G(h)} G(a)$$

*is an isomorphism for every  $(a, h) \in \mathbb{A}_{\mathcal{T}}$ .*

**4.2. Remark.** It follows from [14, Theorem 2.12] that the second condition of Theorem 4.1 is equivalent to saying that the composite

$$TT(a) \xrightarrow{T(g_{T(a)})} TGT(a) \xrightarrow{\lambda_{T(a)}} GTT(a) \xrightarrow{G(m_a)} GT(a)$$

is an isomorphism for every  $a \in \mathbb{A}$ .

## 5. COMPATIBLE ENTWININGS

Let  $\underline{\mathcal{H}} = (H, m, e)$  be a monad,  $\overline{\mathcal{H}} = (H, \delta, \varepsilon)$  a comonad on  $\mathbb{A}$ , and let  $\lambda : HH \rightarrow HH$  be an entwining from the monad  $\underline{\mathcal{H}}$  to the comonad  $\overline{\mathcal{H}}$ . The datum  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  is called a *monad-comonad triple*. The objects of the category  $\mathbb{A}_{\underline{\mathcal{H}}}^{\overline{\mathcal{H}}}(\lambda)$  are called (*mixed*)  $\lambda$ -bimodules.

**5.1. Lemma.** *The triple  $(H(a), m_a, \delta_a)$  is a  $\lambda$ -bimodule for all  $a \in \mathbb{A}$  if and only if we have a commutative diagram*

$$(5.1) \quad \begin{array}{ccccc} HH & \xrightarrow{m} & H & \xrightarrow{\delta} & HH \\ H\delta \downarrow & & & & \uparrow Hm \\ HHH & \xrightarrow{\lambda H} & HHH & & \end{array}$$

In this case, there are functors

- (1)  $K : \mathbb{A} \rightarrow (\mathbb{A}_{\underline{\mathcal{H}}})^{\widehat{\overline{\mathcal{H}}}}$ ,  $a \mapsto ((H(a), m_a), \delta_a)$ , satisfying  $\phi_{\underline{\mathcal{H}}} = U^{\widehat{\overline{\mathcal{H}}}} K$ ;
- (2)  $K' : \mathbb{A} \rightarrow (\mathbb{A}^{\overline{\mathcal{H}}})_{\widehat{\underline{\mathcal{H}}}}$ ,  $a \mapsto ((H(a), \delta_a), m_a)$ , satisfying  $U_{\widehat{\underline{\mathcal{H}}}} K' = \phi^{\overline{\mathcal{H}}}$ .

**5.2. Definitions.** Given a monad-comonad triple  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ , the entwining  $\lambda : HH \rightarrow HH$  is said to be *compatible* provided Diagram (5.1) is commutative; then  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  is said to be a *compatible monad-comonad triple*.

The triple  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  is called a *bimonad* if it is a compatible triple with additional commutative diagrams (see [13, Definition 4.1])

$$(5.2) \quad \begin{array}{ccc} HH \xrightarrow{H\varepsilon} H & 1 \xrightarrow{e} H & 1 \xrightarrow{e} H \\ m \downarrow \quad (i) \quad \downarrow \varepsilon & e \downarrow \quad (ii) \quad \downarrow \delta & \quad \quad \downarrow \varepsilon \\ H \xrightarrow{\varepsilon} 1, & H \xrightarrow{He} HH, & = \quad \quad \downarrow \varepsilon \\ & & 1. \end{array}$$

Notice that for any monad-comonad triple  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ , to say that Diagram (5.2)(i) commutes is to say that  $\varepsilon : H \rightarrow 1$  is an augmentation of the monad  $\underline{\mathcal{H}}$ , while to say that Diagram (5.2)(ii) commutes is to say that  $e : 1 \rightarrow H$  is a grouplike morphism of the comonad  $\overline{\mathcal{H}}$ . Thus, for any bimonad  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ ,  $e$  is a grouplike morphism of the comonad  $\overline{\mathcal{H}}$  and  $\varepsilon$  is an augmentation of the monad  $\underline{\mathcal{H}}$ .

**5.3. Proposition.** *Let  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  be a compatible monad-comonad triple. If  $\delta \cdot e = He \cdot e$  (i.e.  $e : 1 \rightarrow H$  is a grouplike morphism of  $\overline{\mathcal{H}}$ ), then  $\delta = \lambda \cdot He$  and the comparison functor  $K$  in Lemma 5.1 is induced by the grouplike morphism  $e$ , that is  $K = K_e$ .*

**Proof.** Assume that  $\delta \cdot e = He \cdot e$  and that  $\lambda$  is compatible. Then, in the diagram

$$\begin{array}{ccccccc} H & \xrightarrow{He} & HH & \xrightarrow{m} & H & \xrightarrow{\delta} & HH \\ He \downarrow & & \downarrow H\delta & & & & \uparrow Hm \\ HH & \xrightarrow{HHe} & HHH & \xrightarrow{\lambda H} & HHH & & \\ & \searrow \lambda & & & \nearrow HHe & & \\ & & HH & & & & \end{array},$$



the rectangles commute. Since the triangle is also commutative by naturality of composition and since  $m \cdot He = 1$ , it follows that  $\delta = \lambda \cdot He$ . From Section 4 and (4.1), we conclude that the comparison functor  $K$  is induced by the grouplike morphism  $e$ , that is  $K = K_e$ .  $\square$

**5.4. Remark.** Note that if  $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$  (i.e.  $\varepsilon : H \rightarrow 1$  is an augmentation of  $\underline{\mathcal{H}}$ ) and  $\lambda$  is compatible, then postcomposing the diagram (5.1) with the morphism  $H\varepsilon$  implies  $m = H\varepsilon \cdot \lambda$ .

In the next propositions we do not require a priori  $\lambda$  to be a compatible entwining.

**5.5. Proposition.** *Let  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  be a monad-comonad triple.*

- (i) *If  $\delta = \lambda \cdot He$ , then  $\delta \cdot e = He \cdot e$ ;*
- (ii) *if  $m = H\varepsilon \cdot \lambda$ , then  $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$ .*

*Moreover, if one of these conditions is satisfied, then  $\varepsilon \cdot e = 1$ , provided that  $e : 1 \rightarrow H$  is a (componentwise) monomorphism or  $\varepsilon$  is a (componentwise) epimorphism.*

**Proof.** (i) Assume  $\delta = \lambda \cdot He$ . Since  $He \cdot e = eH \cdot e$  (by naturality) and  $\lambda \cdot eH = He$  (see 3.1),

$$\delta \cdot e = \lambda \cdot He \cdot e = \lambda \cdot eH \cdot e = He \cdot e.$$

- (ii) Assume  $m = H\varepsilon \cdot \lambda$ . Since  $\varepsilon \cdot H\varepsilon = \varepsilon \cdot \varepsilon H$  and  $\varepsilon H \cdot \lambda = H\varepsilon$  (see 3.1),

$$\varepsilon \cdot m = \varepsilon \cdot H\varepsilon \cdot \lambda = \varepsilon \cdot H\varepsilon \cdot \lambda = \varepsilon \cdot H\varepsilon.$$

To show the final claim, observe that  $\delta = \lambda \cdot He$  implies

$$1 = \varepsilon H \cdot \delta = \varepsilon H \cdot \lambda \cdot He = H\varepsilon \cdot He,$$

and  $m = H\varepsilon \cdot \lambda$  implies

$$1 = m \cdot eH = H\varepsilon \cdot \lambda \cdot eH = H\varepsilon \cdot He,$$

so in both cases,  $1 = H\varepsilon \cdot He$ . Naturality of  $e$  and  $\varepsilon$  imply commutativity of the diagrams, respectively,

$$\begin{array}{ccc} H & \xrightarrow{eH} & HH \\ \varepsilon \downarrow & & \downarrow H\varepsilon \\ 1 & \xrightarrow{e} & H, \end{array} \quad \begin{array}{ccc} H & \xrightarrow{He} & HH \\ \varepsilon \downarrow & & \downarrow \varepsilon H \\ 1 & \xrightarrow{e} & H. \end{array}$$

From the left hand diagram one gets

$$e = H\varepsilon \cdot He \cdot e = H\varepsilon \cdot eH \cdot e = e \cdot \varepsilon \cdot e,$$

thus if  $e$  is a (componentwise) monomorphism,  $\varepsilon \cdot e = 1$ , while the right hand diagram implies

$$\varepsilon = \varepsilon \cdot H\varepsilon \cdot He = \varepsilon \cdot e \cdot \varepsilon$$

and hence  $\varepsilon \cdot e = 1$  provided  $\varepsilon$  is a (componentwise) epimorphism.  $\square$

**5.6. Lemma.** *Let  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  be a monad-comonad triple. If*

$$m = H\varepsilon \cdot \lambda \quad \text{or} \quad \delta = \lambda \cdot He,$$

*then  $\lambda$  is compatible, that is, diagram (5.1) is commutative.*

**Proof.** If  $\delta = \lambda \cdot He$ , the triangle is commutative in the diagram

$$\begin{array}{ccccc} HH & \xrightarrow{H\delta} & HHH & \xrightarrow{\lambda H} & HHH \\ HH\downarrow e & \nearrow H\lambda & & & \downarrow Hm \\ HHH & \xrightarrow{mH} & HH & \xrightarrow{\lambda} & HH, \end{array}$$

whereas the trapezium is commutative by the entwining property of  $\lambda$ . The left path of the outer diagram is

$$\lambda \cdot mH \cdot HH\downarrow e = \lambda \cdot He \cdot m = \delta \cdot m.$$

This shows that (5.1) is commutative.

In a similar way the claim for  $m = H\varepsilon \cdot \lambda$  is proved.  $\square$

To sum up, combining Proposition 5.5, Remark 5.4 and Lemma 5.6 yields

**5.7. Proposition.** *Let  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  be a monad-comonad triple.*

- (1)  $\delta = \lambda \cdot He$  if and only if  $\lambda$  is compatible and  $\delta \cdot e = He \cdot e$ ;
- (2)  $m = H\varepsilon \cdot \lambda$  if and only if  $\lambda$  is compatible and  $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$ ;
- (3) if  $\delta = \lambda \cdot He$ ,  $m = H\varepsilon \cdot \lambda$ , and  $e : 1 \rightarrow H$  is a (componentwise) monomorphism or  $\varepsilon$  is a (componentwise) epimorphism, then  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  is a bimonad (see 5.2).

If  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  is a monad-comonad triple such that  $\delta = \lambda \cdot He$ , then  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  is a compatible monad-comonad triple by Lemma 5.6, and hence, by Proposition 5.3, the assignment  $a \mapsto (H(a), m_a, \delta_a)$  yields the functor  $K_e : \mathbb{A} \rightarrow \mathbb{A}_{\underline{\mathcal{H}}}^{\overline{\mathcal{H}}}(\lambda)$  with commutative diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{K_e} & \mathbb{A}_{\underline{\mathcal{H}}}^{\overline{\mathcal{H}}}(\lambda) = (\mathbb{A}_{\underline{\mathcal{H}}})^{\widehat{\overline{\mathcal{H}}}} \\ & \searrow \phi_{\underline{\mathcal{H}}} & \downarrow U^{\widehat{\overline{\mathcal{H}}}} \\ & & \mathbb{A}_{\underline{\mathcal{H}}}. \end{array}$$

Recall from [13] that a bimonad  $\mathcal{H}$  is said to be a *Hopf monad* provided it has an *antipode*, i.e. there exists a natural transformation  $S : H \rightarrow H$  such that  $m \cdot HS \cdot \delta = e \cdot \varepsilon = m \cdot SH \cdot \delta$ .

**5.8. Theorem.** *Let  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  be a monad-comonad triple on a Cauchy complete category  $\mathbb{A}$ . Assume that  $\delta = \lambda \cdot He$  and  $e : 1 \rightarrow H$  is a (componentwise) monomorphism. Then the following are equivalent:*

- (a)  $K_e : \mathbb{A} \rightarrow \mathbb{A}_{\underline{\mathcal{H}}}^{\overline{\mathcal{H}}}(\lambda)$  is an equivalence of categories;

(b) the composite  $H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{H(h)} H(a)$  is an isomorphism for every  $(a, h) \in \mathbb{A}_{\underline{\mathcal{H}}}$ ;

(c) the composite  $HH \xrightarrow{\delta H} HHH \xrightarrow{Hm} HH$  is an isomorphism.

If, in addition,  $\varepsilon : H \rightarrow 1$  is an augmentation of the monad  $\underline{\mathcal{H}}$ , then  $\mathcal{H}$  is a Hopf monad.

**Proof.** Since  $\delta = \lambda \cdot He$ , (a)  $\Rightarrow$  (b) is trivial by Theorem 4.1, while (b) and (c) are equivalent by Remark 4.2.

Given (c), it follows from Theorem 4.1 that  $K$  is an equivalence of categories if and only if the functor  $\phi_{\underline{\mathcal{H}}}$  is comonadic. But by [11, Corollary 3.17] this is always the case, since  $e : 1 \rightarrow H$  is a monomorphism and hence  $\varepsilon \cdot e = 1$  by Proposition 5.5. This proves the implication (c)  $\Rightarrow$  (a).

Finally, if  $\varepsilon : H \rightarrow 1$  is an augmentation of the monad  $\underline{\mathcal{H}}$ , then  $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$ , and since  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  is compatible,  $m = H\varepsilon \cdot \lambda$  by Proposition 5.7. Since  $\delta = \lambda \cdot He$ ,  $\delta \cdot e = He \cdot e$  again by Proposition 5.7. Thus  $\mathcal{H}$  is a bimonad and it now follows from [14, 3.1] that  $\mathcal{H}$  is a Hopf monad.  $\square$

## 6. GENERALISED BIALGEBRAS

In this section, we apply our results in the context of operads to recover results of Loday on generalised bialgebras in [7]. The Leitmotiv of the section is that a (co)operad is a particular type of (co)monad. Let  $\mathbf{k}$  denote a field and  $\mathbb{A}$  the category of  $\mathbf{k}$ -vector spaces.

**6.1. Schur functors.** An  $\mathbb{S}$ -module  $\mathcal{M}$  in  $\mathbb{A}$  (or *vector species*) is a collection of objects  $\mathcal{M}(n)$ , for  $n > 0$ , together with an action of the symmetric group  $S_n$ . To an  $\mathbb{S}$ -module  $\mathcal{M}$  one associates the functor

$$\begin{aligned} F_{\mathcal{M}} : \quad \mathbb{A} &\longrightarrow \mathbb{A} \\ V &\mapsto \bigoplus_{n>0} \mathcal{M}(n) \otimes_{\mathbf{k}[S_n]} V^{\otimes n} . \end{aligned}$$

Such a functor is called a *Schur functor*. Joyal proved in [6] that for two  $\mathbb{S}$ -modules  $\mathcal{M}$  and  $\mathcal{N}$ , the composite  $F_{\mathcal{M}} \cdot F_{\mathcal{N}}$  is a Schur functor of the form  $F_{\mathcal{M} \circ \mathcal{N}}$  with  $\mathcal{M} \circ \mathcal{N}$  being the  $\mathbb{S}$ -module defined by

$$(\mathcal{M} \circ \mathcal{N})(n) = \bigoplus_{k>0, i_1+\dots+i_k=n} \mathcal{M}(k) \otimes_{\mathbf{k}[S_k]} \text{Ind}_{S_{i_1} \times \dots \times S_{i_k}}^{S_n} \mathcal{N}(i_1) \otimes \dots \otimes \mathcal{N}(i_k).$$

The product  $\circ$  is called the *plethysm* of  $\mathbb{S}$ -modules, and the category of  $\mathbb{S}$ -modules, together with the plethysm is a monoidal category. The unit for the plethysm is the  $\mathbb{S}$ -module

$$1(n) = \begin{cases} \mathbf{k}, & \text{if } n = 1, \\ 0, & \text{else.} \end{cases}$$

For our purpose, we will always assume that any  $\mathbb{S}$ -module  $\mathcal{M}$  satisfies  $\mathcal{M}(1) = \mathbf{k}$ .

We denote by  $e_{\mathcal{M}} : 1 \rightarrow F_{\mathcal{M}}$  the natural transformation which maps  $V$  to the summand  $V$  of  $F_{\mathcal{M}}(V)$  and by  $\varepsilon_{\mathcal{M}} : F_{\mathcal{M}} \rightarrow 1$  the projection of  $F_{\mathcal{M}}(V)$  to the summand  $V$ . Then  $\varepsilon_{\mathcal{M}} \cdot e_{\mathcal{M}} = 1$ .

**6.2. Operads, cooperads.** An operad  $\mathcal{A}$  in  $\mathbb{A}$  is a monoid in the monoidal category of  $\mathbb{S}$ -modules. This amounts to say that the functor  $F_{\mathcal{A}}$  is the functor part of a monad  $\mathcal{T}_{\mathcal{A}} = (F_{\mathcal{A}}, m_{\mathcal{A}}, e_{\mathcal{A}})$ .

An algebra over an operad  $\mathcal{A}$ , or  $\mathcal{A}$ -algebra, is a  $\mathcal{T}_{\mathcal{A}}$ -module. Hence, the free  $\mathcal{A}$ -algebra generated by a vector space  $V$  is nothing else than  $(\mathcal{T}_{\mathcal{A}}(V), (m_{\mathcal{A}})_V, (e_{\mathcal{A}})_V)$ .

A cooperad  $\mathcal{C}$  in  $\mathbb{A}$  is a comonoid in the monoidal category of  $\mathbb{S}$ -modules. This amounts to say that the functor  $F_{\mathcal{C}}$  is the functor part of a comonad  $\mathcal{G}_{\mathcal{C}} = (F_{\mathcal{C}}, \delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ .

A coalgebra over a cooperad  $\mathcal{C}$ , or  $\mathcal{C}$ -coalgebra, is a  $\mathcal{G}_{\mathcal{C}}$ -comodule.

Note that one has to be a little careful with the definition of cooperads if one wants a linear duality between operads and cooperads (see [9]). With our definition and assumptions, any coalgebra over a cooperad  $\mathcal{C}$  is naturally conilpotent.

We assume that, for any  $\mathbb{S}$ -module  $\mathcal{M}$ , the  $\mathbf{k}$ -vector space  $\mathcal{M}(n)$  is finite dimensional. We assume also that either the action of the symmetric group is free or the field  $\mathbf{k}$  has characteristic 0.

**6.3. Proposition.** *If  $\mathcal{A}$  is an operad, then  $\varepsilon_{\mathcal{A}}$  is an augmentation for the monad  $\mathcal{T}_{\mathcal{A}}$ . If  $\mathcal{C}$  is a cooperad then  $e_{\mathcal{C}}$  is a grouplike morphism for the comonad  $\mathcal{G}_{\mathcal{C}}$ .*

**Proof.** The unit for the plethysm forms a (co)operad and the associated (co)monad is the identity functor. Let  $m : \mathcal{A} \circ \mathcal{A} \rightarrow \mathcal{A}$  denote the operad composition. One has to prove that, for every  $n \geq 1$ , the following diagram is commutative:

$$\begin{array}{ccc} (\mathcal{A} \circ \mathcal{A})(n) & \xrightarrow{\mathcal{A} \circ \varepsilon_{\mathcal{A}}} & \mathcal{A}(n) \\ m \downarrow & & \downarrow \varepsilon_{\mathcal{A}} \\ \mathcal{A}(n) & \xrightarrow{\varepsilon_{\mathcal{A}}} & 1(n). \end{array}$$

If  $n > 1$ , then the diagram commutes because the top and bottom compositions vanish. If  $n = 1$ , since  $\mathcal{A}(1) = \mathbf{k}$  then  $(\mathcal{A} \circ \mathcal{A})(1) = \mathbf{k} \otimes \mathbf{k} = \mathbf{k}$  and  $m$  is the identity as well as  $\mathcal{A} \circ \varepsilon_{\mathcal{A}}$  and  $\varepsilon_{\mathcal{A}}$ . So the diagram is commutative. Furthermore, we have seen in Section 6.1 that  $\varepsilon_{\mathcal{A}} \cdot e_{\mathcal{A}} = 1$ . A similar proof shows that  $e_{\mathcal{C}}$  is a grouplike morphism for the comonad  $\mathcal{G}_{\mathcal{C}}$ .  $\square$

**6.4. Distributive laws and generalised bialgebras.** Let  $\mathcal{A}$  be an operad and  $\mathcal{C}$  be a cooperad.

**(H0)** A distributive law between  $\mathcal{A}$  and  $\mathcal{C}$  is a morphism of  $\mathbb{S}$ -modules  $\mathcal{A} \circ \mathcal{C} \rightarrow \mathcal{C} \circ \mathcal{A}$  satisfying some relations which amount to say that the corresponding natural transformation

$$\lambda : F_{\mathcal{A} \circ \mathcal{C}} = F_{\mathcal{A}} F_{\mathcal{C}} \longrightarrow F_{\mathcal{C}} F_{\mathcal{A}} = F_{\mathcal{C} \circ \mathcal{A}}$$

is an entwining. If such an entwining exists, we say, as in [7], that *hypothesis (H0) is satisfied*. Under this hypothesis, an object  $(V, h, \theta)$  in  $(\mathbb{A}_{\mathcal{T}_{\mathcal{A}}})^{\widehat{\mathcal{G}}_{\mathcal{C}}}$  is called a  $(\mathcal{C}, \mathcal{A})$ -bialgebra.

**(H1)** Assume that there is a map  $\alpha : \mathcal{A} \rightarrow \mathcal{C} \circ \mathcal{A}$  making  $\mathcal{A}$  a left  $\mathcal{C}$ -comodule, that is, every free  $\mathcal{A}$ -algebra is endowed with a structure of a  $\mathcal{C}$ -coalgebra. This amounts to say that there is a functor  $K : \mathbb{A} \rightarrow (\mathbb{A}_{\mathcal{T}_{\mathcal{A}}})^{\widehat{\mathcal{G}}_{\mathcal{C}}}$  such that the diagram (3.1) is commutative. If such a functor exists, we say, as in [7], that *hypothesis (H1) is satisfied*. The corresponding left  $\mathcal{G}_{\mathcal{C}}$ -comodule structure on  $\mathcal{T}_{\mathcal{A}}$  is given by  $\alpha : F_{\mathcal{A}} \rightarrow F_{\mathcal{C}}F_{\mathcal{A}}$ .

At the level of  $\mathbb{S}$ -modules one gets that  $\alpha_1 : \mathcal{A}(1) = \mathbf{k} \rightarrow (\mathcal{C} \circ \mathcal{A})(1) = \mathbf{k}$  is the identity, because  $(\varepsilon_{\mathcal{C}} \circ \mathcal{A}) \cdot \alpha = 1$ , so that

$$\alpha \cdot e_{\mathcal{A}} = e_{\mathcal{C}}F_{\mathcal{A}} \cdot e_{\mathcal{A}} = F_{\mathcal{C}}e_{\mathcal{A}} \cdot e_{\mathcal{C}}.$$

Thus, the diagram

$$(6.1) \quad \begin{array}{ccccc} & & F_{\mathcal{A}} & & \\ & \nearrow e_{\mathcal{A}} & & \searrow \alpha & \\ 1 & \xrightarrow{e_{\mathcal{A}}} & F_{\mathcal{A}} & \xrightarrow{e_{\mathcal{C}}T} & F_{\mathcal{C}}F_{\mathcal{A}} \\ & \searrow e_{\mathcal{C}} & & \nearrow F_{\mathcal{C}}e_{\mathcal{A}} & \\ & & F_{\mathcal{C}} & & \end{array}$$

commutes. As a consequence, if  $(V, h) \in \mathbb{A}_{\mathcal{T}_{\mathcal{A}}}$ , then

$$F_{\mathcal{C}}(h) \cdot \alpha_V \cdot (e_{\mathcal{A}})_V = F_{\mathcal{C}}(h) \cdot (F_{\mathcal{C}}e_{\mathcal{A}})_V \cdot (e_{\mathcal{C}})_V = (e_{\mathcal{C}})_V,$$

and since the  $(V, h)$ -component  $\beta_{(V, h)}$  of the right  $\mathcal{G}_{\mathcal{C}}$ -comodule structure on  $\beta : U_{\mathcal{T}_{\mathcal{A}}} \rightarrow U_{\mathcal{T}_{\mathcal{A}}}\widehat{F_{\mathcal{C}}}$  is just the composite  $F_{\mathcal{C}}(h) \cdot \alpha_V \cdot (e_{\mathcal{A}})_V$ , we get

$$(6.2) \quad \beta_{(V, h)} = (e_{\mathcal{C}})_V.$$

Thus,  $\beta$  is defined by the grouplike morphism  $e_{\mathcal{C}} : 1 \rightarrow F_{\mathcal{C}}$  and hence the comparison functor  $K : \mathbb{A} \rightarrow (\mathbb{A}_{\mathcal{T}_{\mathcal{A}}})^{\widehat{\mathcal{G}}_{\mathcal{C}}}$  is induced by this grouplike morphism, i.e.  $K = K_{e_{\mathcal{C}}}$ . So we can apply the results of the previous sections to the present setting, in particular, (4.1) gives

$$(6.3) \quad \alpha = \lambda \cdot F_{\mathcal{A}}e_{\mathcal{C}}.$$

We assume that the hypotheses (H0) and (H1) hold. Consider the  $\mathcal{C}$ -comodule map  $\varphi : \mathcal{A} \rightarrow \mathcal{C}$  induced by the projection  $\varepsilon_{\mathcal{A}} : \mathcal{A} \rightarrow 1$ . Since  $\varphi = (\mathcal{C} \circ \varepsilon_{\mathcal{A}}) \cdot \alpha$ , where  $\alpha : \mathcal{A} \rightarrow \mathcal{C} \circ \mathcal{A}$  is the  $\mathcal{C}$ -comodule morphism of hypothesis (H1), one has, for every  $\mu \in \mathcal{A}(n)$ ,

$$(6.4) \quad \alpha(\mu) = \varphi_n(\mu) \otimes 1^{\otimes n} + \sum_{k < n} c_k^{\mu} \otimes \alpha_1^{\mu} \otimes \dots \otimes \alpha_k^{\mu},$$

where  $\varphi_n$  is the component of  $\varphi$  on  $\mathcal{A}(n)$ ,  $c_k^{\mu} \in \mathcal{C}(k)$ , and  $\alpha_i^{\mu} \in \mathcal{A}(l_i)$  with  $\sum_{i=1}^k l_i = n$ .

**(H2iso)** When  $\varphi$  is an isomorphism, we say, as in [7], that *hypothesis (H2iso) is satisfied*.

In the sequel we will be interested in the link between  $\varphi$  and the comonad morphism  $t : \phi_{\mathcal{T}_{\mathcal{A}}} U_{\mathcal{T}_{\mathcal{A}}} \longrightarrow \widehat{F}_{\mathcal{C}}$  as in section 2.3. Recall that for every  $(V, h) \in \mathbb{A}_{\mathcal{T}_{\mathcal{A}}}$ ,  $t_{(V, h)}$  is the composite

$$F_{\mathcal{A}}(V) \xrightarrow{\alpha_V} F_{\mathcal{C}} F_{\mathcal{A}}(V) \xrightarrow{F_{\mathcal{C}} h} F_{\mathcal{C}}(V).$$

**6.5. Lemma.** *Assume the hypotheses (H0) and (H1). Then the map  $\varphi$  is an isomorphism if and only if  $t$  is an isomorphism.*

**Proof.** We use the natural arity-grading on  $\mathbb{S}$ -modules. Given  $\mu \in \mathcal{A}(n), \underline{v} \in V^{\otimes n}$ , one has

$$t_{(V, h)}(\mu \otimes \underline{v}) = \varphi_n(\mu) \otimes \underline{v} + \sum_{k < n} c_k^\mu \otimes \alpha_1^\mu(\underline{v}_1) \otimes \dots \otimes \alpha_k^\mu(\underline{v}_k),$$

where  $\underline{v} = \underline{v}_1 \otimes \dots \otimes \underline{v}_k \in V^{\otimes l_1} \otimes \dots \otimes V^{\otimes l_k}$ . This is a triangular system with dominant coefficient  $\varphi_n$ . As a consequence, we get that if  $\varphi$  is an isomorphism so is  $t_{(V, h)}$ . The converse is immediate because  $\varphi_V = t_{(V, (\varepsilon_{\mathcal{A}})_V)}$  for all  $V \in \mathbb{A}$ .  $\square$

**6.6. The primitive part of a  $(\mathcal{C}, \mathcal{A})$ -bialgebra.** Because the category of  $\mathbf{k}$ -vector spaces admits equalisers, under the hypotheses (H0) and (H1), the functor  $K$  admits a right adjoint  $R$  whose value at  $((H, h), \theta) \in (\mathbb{A}_{\mathcal{T}_{\mathcal{A}}})^{\widehat{\mathcal{G}_{\mathcal{C}}}}$  appears as the equaliser

$$R((H, h), \theta) \xrightarrow{i_{((H, h), \theta)}} H \xrightarrow[(e_{\mathcal{C}})_H]{\theta} \mathcal{C}(H).$$

As a consequence,

$$R((H, h), \theta) = \{x \in H, \theta(x) = 1 \otimes x\},$$

and thus  $R((H, h), \theta)$  is just the *primitive part*  $\text{Prim}V$  of the  $(\mathcal{C}, \mathcal{A})$ -bialgebra  $(H, h, \theta)$  in the sense of Loday [7].

We are now in the position to state and prove our main result.

**6.7. Rigidity Theorem.** ([7, Theorem 2.3.7]) *Let  $\mathcal{A}$  be an operad,  $\mathcal{C}$  a cooperad, and  $\mathcal{T}_{\mathcal{A}} = (F_{\mathcal{A}}, m, e_{\mathcal{A}})$  and  $\mathcal{G}_{\mathcal{C}} = (F_{\mathcal{C}}, \delta, \varepsilon_{\mathcal{C}})$  the corresponding monad and comonad on  $\mathbb{A}$ . Suppose that the hypotheses (H0), (H1) and (H2iso) are fulfilled. Then the comparison functor*

$$K_{e_{\mathcal{C}}} : \mathbb{A} \longrightarrow (\mathbb{A}_{\mathcal{T}_{\mathcal{A}}})^{\widehat{\mathcal{G}_{\mathcal{C}}}}$$

*is an equivalence of categories. Hence, in particular, any  $(\mathcal{C}, \mathcal{A})$ -bialgebra  $(H, h, \theta)$  is a free  $\mathcal{A}$ -algebra and a cofree conilpotent  $\mathcal{C}$ -coalgebra generated by  $\text{Prim}H$ .*

**Proof.** Because the hypothesis (H2iso) is satisfied, it follows from Lemma 6.5 that  $t_{(V,h)}$  is an isomorphism for all  $(V,h) \in \mathbb{A}_{F_{\mathcal{A}}}$ . Moreover, since  $\varepsilon_{\mathcal{A}} \cdot e_{\mathcal{A}} = 1$ , and since  $\mathbb{A}$  is clearly Cauchy complete, the functor  $\phi_{\mathcal{T}_{\mathcal{A}}} : \mathbb{A} \rightarrow \mathbb{A}_{\mathcal{T}_{\mathcal{A}}}$  is comonadic by [11, Corollary 3.17]. Applying now Theorem 4.1, we get the result.  $\square$

**6.8. Remark.** In [7], for the proof of this theorem, Loday builds idempotents to produce a projection onto the primitive part. An advantage of our proof is that it does not need such a construction.

The following corollary is a special case of the Rigidity Theorem, where it is not necessary to verify hypothesis (H2iso).

**6.9. Corollary.** *Let  $\mathcal{M}$  be an  $\mathbb{S}$ -module carrying a structure of an operad  $\mathcal{A} = (\mathcal{M}, m, e_{\mathcal{M}})$ , a structure of cooperad  $\mathcal{C} = (\mathcal{M}, \delta, \varepsilon_{\mathcal{M}})$ , and let*

$$\lambda : \mathcal{M} \circ \mathcal{M} \rightarrow \mathcal{M} \circ \mathcal{M}$$

*be an entwining between  $\mathcal{A}$  and  $\mathcal{C}$ . If one of the three equivalent conditions*

$$(i) \lambda \text{ is compatible, } (ii) \delta = \lambda \cdot (\mathcal{M} \circ e_{\mathcal{M}}), \quad (iii) m = (\mathcal{M} \circ \varepsilon_{\mathcal{M}}) \cdot \lambda,$$

*holds, then the compatible monad-comonad triple  $(\mathcal{T}_{\mathcal{A}}, \mathcal{G}_{\mathcal{C}}, \lambda)$  is a Hopf monad. Moreover, any  $(\mathcal{C}, \mathcal{A})$ -bialgebra is a free  $\mathcal{A}$ -algebra and a cofree conilpotent  $\mathcal{C}$ -coalgebra.*

**Proof.** Let us denote by  $\mathcal{H}$  the monad-comonad triple  $(\mathcal{T}_{\mathcal{A}}, \mathcal{G}_{\mathcal{C}}, \lambda)$ . By Proposition 6.3, the triple satisfies Relations (5.2), and since  $e_{\mathcal{M}}$  is a componentwise monomorphism,  $\mathcal{H}$  is a bimonad by Proposition 5.7. Thus there is a comparison functor

$$K : \mathbb{A} \rightarrow (\mathbb{A}_{\mathcal{T}_{\mathcal{A}}})^{\widehat{\mathcal{G}_{\mathcal{C}}}}, \quad V \mapsto ((\mathcal{M}(V), m_V), \delta_V),$$

and  $K = K_{e_{\mathcal{M}}}$ .

We can apply Theorem 5.8 to conclude that the functor  $K$  is an equivalence of categories if and only if the composite

$$\mathcal{M}(V) \xrightarrow{\delta_V} (\mathcal{M} \circ \mathcal{M})(V) \xrightarrow{\mathcal{M}(h)} \mathcal{M}(V)$$

is an isomorphism for every  $(V,h) \in \mathbb{A}_{\mathcal{T}_{\mathcal{A}}}$ . But  $\mathcal{M}(h) \cdot \delta_V = t_{(V,h)}$ , where  $t_{(V,h)}$  is the  $(V,h)$ -component of the comonad morphism  $t : \phi_{\mathcal{T}_{\mathcal{A}}} U_{\mathcal{T}_{\mathcal{A}}} \rightarrow \widehat{\mathcal{G}_{\mathcal{C}}}$  induced by  $K$ . It follows that  $\varphi_V = t_{(V, \varepsilon_V)} = \mathcal{M}(\varepsilon_V) \cdot \delta_V = 1$  for every  $V \in \mathbb{A}$ . Thus  $\varphi$  is an isomorphism and then  $t$  is also an isomorphism by Lemma 6.5. Hence  $K$  is an equivalence of categories. It now follows from [14, 3.1] that  $\mathcal{H}$  is a Hopf monad. Furthermore, the Rigidity Theorem applies to our case because (H2iso) is satisfied.  $\square$

**6.10. Example.** As an example we treat the case of infinitesimal bialgebras. Consider the functor  $V \mapsto \mathcal{A}(V) = \bigoplus_n V^{\otimes n}$ . It forms a monad  $\mathcal{T} = (\mathcal{A}, m, e)$  for the concatenation product. One can formulate this as

$$\begin{array}{ccc}
m_V : \mathcal{A}_1 \mathcal{A}_2(V) & \rightarrow & \mathcal{A}(V) \\
\otimes_1 & \mapsto & \otimes \\
\otimes_2 & \mapsto & \otimes \\
v & \mapsto & v
\end{array}
\quad
\begin{array}{ccc}
e_V : V & \rightarrow & \mathcal{A}(V) \\
v & \mapsto & v
\end{array}$$

where  $\mathcal{A}_1$  denotes the “first copy” of  $\mathcal{A}$ . It reads like this: any word in  $\mathcal{A}_1 \mathcal{A}_2(V)$  is composed with letters in  $\{\otimes_1, \otimes_2, v \in V\}$  and the map indicates how it acts on letters.

The functor  $\mathcal{A}$  forms a comonad  $\mathcal{G} = (\mathcal{A}, \delta, \varepsilon)$  with the deconcatenation

$$\begin{array}{ccc}
\delta_V : \mathcal{A}(V) & \rightarrow & \mathcal{A}_1 \mathcal{A}_2(V) \\
\otimes & \mapsto & \otimes_1 + \otimes_2 \\
v & \mapsto & v
\end{array}
\quad
\begin{array}{ccc}
\varepsilon_V : \mathcal{A}(V) & \rightarrow & V \\
\otimes & \mapsto & 0 \\
v & \mapsto & v.
\end{array}$$

The infinitesimal distributive law reads

$$\begin{array}{ccc}
\lambda_V : \mathcal{A}_1 \mathcal{A}_2(V) & \rightarrow & \mathcal{A}_1 \mathcal{A}_2(V) \\
\otimes_1 & \mapsto & \otimes_1 + \otimes_2 \\
\otimes_2 & \mapsto & \otimes_1 \\
v & \mapsto & v.
\end{array}$$

We easily see that  $m$  is associative,  $\delta$  is coassociative, and  $\lambda$  is an entwining.

As an example, we check one of the diagrams for  $\lambda$  (see 3.1):

$$\begin{array}{ccccc}
\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 & \xrightarrow{m_{\mathcal{A}}} & \mathcal{A}_1 \mathcal{A}_2 & & \\
\mathcal{A}\lambda \downarrow & & \downarrow \lambda & & \\
\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 & \xrightarrow{\lambda_{\mathcal{A}}} \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 & \xrightarrow{\mathcal{A}m} & \mathcal{A}_1 \mathcal{A}_2 & .
\end{array}$$

The top arrows send  $\otimes_1 \mapsto \otimes_1 \mapsto \otimes_1 + \otimes_2$ ,  $\otimes_2 \mapsto \otimes_1 \mapsto \otimes_1 + \otimes_2$  and  $\otimes_3 \mapsto \otimes_2 \mapsto \otimes_1$ , while the lower maps send  $\otimes_1 \mapsto \otimes_1 \mapsto \otimes_1 + \otimes_2 \mapsto \otimes_1 + \otimes_2$ ,  $\otimes_2 \mapsto \otimes_2 + \otimes_3 \mapsto \otimes_1 + \otimes_3 \mapsto \otimes_1 + \otimes_2$  and  $\otimes_3 \mapsto \otimes_2 \mapsto \otimes_1 \mapsto \otimes_1$ , which proves commutativity of this diagram.

We have clearly  $\delta = \lambda \cdot \mathcal{A}e$  and  $m = \mathcal{A}\varepsilon \cdot \lambda$ . Consequently, Theorem 6.9 holds. Hereby we recover the Rigidity Theorem of Loday and Ronco for infinitesimal bialgebras which says that any infinitesimal bialgebra is freely and cofreely generated by its primitive part (see [8, Theorem 2.6]).

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